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Efficient Real-Time Routing for Autonomous Vehicles Through Bayes Correlated Equilibrium: An Information Design Framework

Yixuan Liu[1]
yixuan.liu@utexas.edu

Andrew B. Whinston[1,2]
abw@uts.cc.utexas.edu

[1]McCombs School of Business, University of Texas at Austin
[2]Department of Computer Science, University of Texas at Austin

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Abstract

In this paper, we propose an information-based approach to eliminate inefficiency in traffic systems in the era of autonomous vehicles. We build up theoretical models to coordinate vehicles through Waze, a pervasive crowdsourcing mapping app. We apply the idea of Bayesian persuasion (Kamenica and Gentzkow, 2011) in the basic model of a single vehicle and implement the unified information design framework (Bergemann and Morris, 2017) in the general model. Since the reliability of the information source (Waze) is crucial, we also incorporate queueing theory into the congestion model to obtain more accurate predictions of the traffic conditions. We demonstrate the significant efficiency improvement of implementing theoretical economic approach in the robotic area.

Keywords: Bayesian persuasion; information design; autonomous vehicles; Waze

JEL Classification: R41; O35; C72

1 Introduction

Traffic congestion is an inevitable problem for people in urban areas across the world. Economists and transportation researchers have devoted decades to understanding the human behavioral interactions involved and looking for solutions to reduce it. Unfortunately, the usual solution of expanding road system capacity does not always work. As Pigou (1932) and Braess (Braess, Nagurney and Wakolbinger, 2005) have discovered, if every driver chooses the path that looks the most favorable to her, increasing capacity can lead to longer travel time overall. This counterintuitive finding stems from the fact that each driver's selfish decision in the aggregate can lead to a suboptimal social outcome. Roughgarden and Tardos (2002) and Roughgarden (2005) provide a quantitative measure of the inefficiency resulting from drivers selfish routing behavior. They demonstrate that, generally, inefficiency, also known as "the price of anarchy", can be extremely severe. Apparently, the inefficiency of anarchy can be eliminated through coordination. Nevertheless, nowadays, there seems to be no viable tool that can achieve it. However, as technology advances, especially in the era of the Internet of Things (IoT), is there anything people can do to handle the inefficiency of the transportation systems?

When we think about the future of transportation, we cannot ignore the development of autonomous vehicles. Tesla, one of the pioneers of self-driving cars, has attracted exuberant investors through its potential to revolutionize the auto industry. Its stock valuation was thus boosted to surpass Ford and become comparable to that of GM. Other automakers, such as Volkswagen, BMW, and Ford, and tech companies such as Apple, Google, and Uber, are also competing fiercely in the autonomous driving market. It seems that autonomous vehicles are the future of transportation. While the various advantages of autonomous vehicles, including increasing safety and the facilitation of mobility services, are frequently discussed, their effect on improving the efficiency of the traffic systems is rarely mentioned.

In this paper, we propose an information-based approach stemming from game theory to eliminate inefficiency in the era of autonomous vehicles. We believe robotics is an ideal area to apply theoretical economic methods. In fact, though numerous theorems have been derived by economists who study game theory, the theoretical results are hardly applied directly to guiding human behaviors because of their complexness and randomness. However, in the era of the IoT, since robots act as programmed, theories are suitable for determining the optimal robotic behaviors. Thus, to guide the routing decisions for autonomous vehicles, we apply the idea of Bayesian persuasion

(Kamenica and Gentzkow, 2011) and information design (Bergemann and Morris, 2017) through Waze, a pervasive crowdsourcing mapping app for coordinating driverless vehicles.

Under Waze's current practice of providing the same traffic information to all vehicles, negative externalities from vehicles' myopic and selfish decisions can occur, hampering the efficiency of the entire traffic system efficiency. When selecting its route, a vehicle does not usually anticipate other vehicles' strategies or consider its impact on the other vehicles behind it. For instance, when traveling on a highway during rush hours, vehicles are notified by Waze about traffic slowdownS on their route ahead. Given this information, most vehicles decide to make a detour to a local road that currently has no congestion. Consequently, these rerouted vehicles suffer from an overall longer travel time due to the congestion they create on the local route. This example by Pigou (1932) is one of the earliest displays of the suboptimality of selfish routing. Given this issue, the information-based approach we propose provides direction that varies with vehicles. Receiving different traffic data, vehicles find distinctive optimal routes from their perspectives. In this way, they are coordinated through customized information conveyed by Waze.

Since Waze is the basis for the information source, it is important that the information source be

Waze and google update during the drive, though.

reliable. However, a vehicle currently searches for the best route based only on the aggregated traffic information provided by Waze at the time of departure, which may end up with a suboptimal route. Since roads are connected and vehicles can switch from one road to another (especially in urban areas), the traffic situation along a route varies continuously. A route that seems optimal based on Waze's data when a vehicle departs may no longer be optimal after the vehicle has been traveling for a while. A dramatic traffic volume change could occur on some section of the route ahead. For example, when a vehicle departs, it may choose to go on the highway which has less congestion than the local road, as shown by Waze. At the same time, an accident happens nearby. Although the accident seems irrelevant to this vehicle, a number of vehicles deviate from their original routes and merge onto the highway to avoid the accident area. The vehicle therefore experiences a longer travel time than expected. It could have arrived at the destination earlier if Waze had anticipated the traffic volume increase on the highway and had indicated to the vehicle that the local road would be a better choice. In other words, under the current practice, Waze's traffic data become "outdated" and could be misleading if Waze does not prepare for possible future traffic situation changes.

Given the two aforementioned issues, we are motivated to obtain better routing instructions and make Waze a more efficient data source for autonomous vehicles. The primary cause of the issues is that the current Waze does not provide onboard computing systems with *predictive* and *personalized* traffic information to steer vehicles in an optimal fashion. Accordingly, we propose an innovative Waze to improve on the current one in two ways. Waze has very rich individual-level traffic data, which should be exploited for a broader usage. In fact, Waze knows the starting point, destination, and planned route of each vehicle. Based on this, we first suggest that, using the queueing theory, the innovative Waze can calculate the monment that any vehicle arrives at a specific section of a route and can thus estimate and predict the future traffic flow. Hence, Waze will be more advanced, since it has more accurate and predictive traffic information. In addition, Waze will be improved to provide information that varies with each vehicle. Receiving different traffic data, vehicles can be provided with distinctive optimal routes from their own perspectives. They are thus coordinated through customized information conveyed by Waze. Therefore, externality problems such as the Braess' paradox can be eliminated, and the "price of anarchy" is reduced.

The innovative Waze we propose works according to the following procedure. Firstly, just as the current version, Waze collects and clusters inclusive crowdsourced real-time traffic data. The data are then incorporated into the queueing model of the congestion to precisely predicts the traffic. Then, upon several vehicles' setting their destinations and departing, besides current traffic situation, additional personalized information is computed for each vehicle. The information is conveyed to each vehicle by implementing augmented reality, a way to supplement/manipulate autonomous vehicles' onboard computers' vision of the current (or even future) traffic conditions. Since the routing problem is decentralized to be solved by each vehicle, the onboard computer with the vision that uses Waze's information makes decisions in the best interest of that vehicle. After these vehicles have departed and are on the route selected by their own onboard computers, Waze updates its data about these vehicles, as well as the entire traffic system, and then works on newly departing vehicles.[1] The most important step is therefore to solve for the optimal information provided to each vehicle. Thus, in this study, we build up theoretical models and look into the dynamic information design problem for the innovative Waze to minimize the total waiting time of vehicles. To improve the efficiency of the traffic system, the optimal information structure is designed to enable vehicles to foresee the changing traffic conditions and take into account other vehicles on the road.

[1] The advanced Waze is responsible for assisting in the routing problem which is the major decision for a vehicle. It gives "instructions" to autonomous vehicles by providing further information. As for a vehicle's micro level activities, including slowing down at intersections, keeping a safe following distance, turning the vehicle, and reacting in unexpected situations, the onboard steering system will take over.

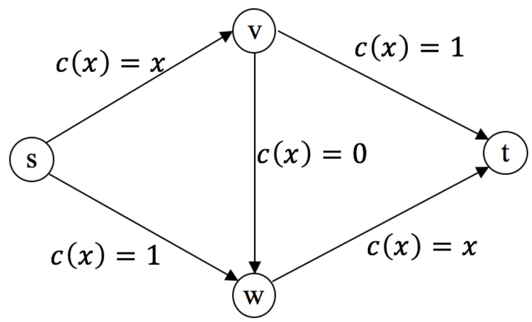

Figure 1: The Braess' Paradox

Before discussing our models in detail, we introduce the following example to illustrate the potential benefits of the innovative Waze proposed in this paper. In the simple four–node traffic network shown in Figure 1, vehicles departs from node s and heads for node t. Each edge is labeled with the cost function $c(\cdot)$ denoting the cost (e.g., travel time) of traveling along the edge as a function of the amount of traffic. Assume a vehicle A is driving along the edge (s, v). The vehicle must then decide to travel either along $v \to w \to t$ or along $v \to t$ when it arrives at node v. Since it is indifferent between the two routes, vehicle A chooses each route with probability $1/2$. If another vehicle, vehicle B, is driving along the edge (s, w) unanticipated by vehicle A, then the expected total cost of the two vehicles is 3. However, if the vehicles are equipped with the innovative Waze, then Waze, foreseeing vehicle B along the edge (s, w), will have vehicle A drive on (v, t). Then, the resulting total cost is 2. The innovative Waze can even give instructions from the beginning, when the two vehicles depart from s. By providing different information to the two vehicles, Waze makes one vehicle travel along $s \to v \to t$ and the other along $s \to w \to t$. The Braess' paradox is thus avoided and the roads are utilized in the best way possible.

To study the real-time traffic information design problem for Waze, we build a discrete-time dynamic programming model. We apply the idea of Bayesian persuasion (Kamenica and Gentzkow, 2011) for the single-vehicle case and apply the unified information design framework of Bergemann and Morris (2017) in a general model. Then, we extend the models to a dynamic setting. The framework can be used to study incomplete information games with a designer who commits to providing information to several players. The information designer's problem is to identify the information structure and equilibrium such that, when the players maximize their ex ante expected utilities, the designer's payoff is maximized. In our setting, since autonomous vehicles (players)[2]

[2]In §2, we introduce examples in which one or two vehicles depart almost at the same time and are thus considered

are robots with or without human beings in them, instead of maximizing their utility functions, we assume that vehicles minimize their expected waiting time. The feasible information structures, defined as Bayes correlated equilibria, satisfy the obedience condition, which requires that, if the designer privately communicates information as stochastic action recommendations, [3] the vehicles will want to follow the recommendations. Among these information structures, Waze (information designer) chooses the one that minimizes the expected total waiting time of all vehicles during the entire rush hour.

Our discrete-time model is more applicable to routing problem on open roads. To consider a more complex traffic system in urban areas, we propose a continuous-time traffic network model. In a city road network, a vehicle can choose from a number of paths that connect the starting point and the destination. We model vehicles departing as a point process (Daley and Vere-Jones, 2007) in which the time interval between two successive vehicles is a random variable with a known distribution. Besides the challenge of selecting the shortest path in a traffic network, the vehicle faces challenges from the highly random nature of future traffic flow along each possible path. A huge volume of data is clustered for Waze. Waze must be able to clean, analyze, and make predictions and recommendations for each vehicle in real-time. The information design problem for urban traffic appears to remain challenging. For the problem to be fully tackled, processors of extremely high computing power need to be onboard for each autonomous vehicle.

More broadly speaking, we study the economic approach of robotic coordination. We emphasize that, in the era of the IoT, normative economic models can be applied to directing robots' behavior. Instead of understanding human behavior, economists may want to take the opportunity to guide optimal robotic actions. For example, besides the routing problem studied in this paper, the information design framework can be applied in many different settings, including the parking problems, autonomous freight trains and trucks, and aircraft take-off and landing control, etc.

In the next section, we start with a static model to present preliminary approach and to illustrate the idea of information design. In Section 3, we study the general dynamic framework of either a single vehicle departing at a time or multiple vehicles departing simultaneously. These two models are more applicable to the problem of intercity or long-distance travel. In Section 4, we study the model of travel in urban areas. In Section 5, we discuss the results and conclude the paper.

together. In §3, we examine a generalized model in which any number of vehicles depart within a short predefined time interval.

[3] A stochastic action recommendation is an actual recommendation generated by an optimized random generator. It is well known in game theory and is rationalized by expected utility theory.

2 Static models

We consider the simple example of two routes from a given starting point to a given destination. Route 1 is on the highways and Route 2 is on a frontage road. In addition, other vehicles can merge onto Route 1 as illustrated in Figures 2 and 3. Congestions can occur on either road. The capacity of route j is assumed to be s_j, $s_1 > s_2$.

A finite set of I autonomous vehicles are departing simultaneously, all heading for their individual destinations. Each vehicle can choose between Route 1 and Route 2. We write $A_i = \{\text{Route } 1, \text{Route } 2\}$ for the action set of vehicle i. Upon departure, the vehicle is informed about the current traffic situation on each road.[4] Specifically, if there is congestion, Waze tells each vehicle the number of vehicles in the queue on route j, $D_j, j = 1, 2$. However, departing vehicles are not sure whether merging traffic will be joining the queue in front of them. Let θ be the state of the merging traffic,

$$\theta = \begin{cases} \lambda, & \text{with probability } \psi(\lambda) \\ 0, & \text{with probability } \psi(0) \end{cases}$$

That is, with probability $\psi(\lambda)$, λ vehicles merge onto the highway in front of departing vehicles. Let $\Theta = \{0, \lambda\}$ be the set of possible states and let $\psi \in \Delta(\Theta)$ define the common prior belief of vehicles about the traffic state.

The utility function of vehicle i is defined as $u_i : A \times \Theta \to \mathbb{R}$. Since the prior distribution is uniquely specified by ψ, with a little abuse of notation, we define the basic game as $G = ((A_i, u_i)_{i=1}^I, \Theta, \psi)$.

The information designer wants to find a decision rule $\sigma : \Theta \to \Delta(A)$ that minimizes the total waiting time of the vehicles. A decision rule is basically a distribution over actions for each traffic state. In a symmetric setting (the basic game and the objective of the information designer are symmetric), we can focus on symmetric decision rules. We define $\sigma(a|\theta)$ as the probability of taking action a given the state θ. Then, for a vehicle obeying recommendations, the ex ante distribution

[4] We make this assumption because the owner of a vehicle can decide whether to depart or not based on the traffic volume. An alternative assumption is that the vehicle has no information about the traffic upon departure and relies fully on the specifically designed information later provided by Waze. That is, Waze has complete control over the vehicle. However, making the alternative assumption will strengthen our results. That is, Waze can make the traffic system even more efficient.

over states and action is given by $\sigma(a|\theta)\psi(\theta)$, and the vehicle's belief is updated by Bayes' rule:

$$\frac{\sigma(a|\theta)\psi(\theta)}{\sum_{\theta}\sigma(a|\theta)\psi(\theta)}.$$

A decision rule is a **Bayes correlated equilibrium** (BCE) if it satisfies the following obedience condition: A player has no incentive to deviate from the recommendation given by the information designer. This condition defines a set of linear constraints. Subject to these constraints, the information designer maximizes welfare or profit. As shown by Bergemann and Morris (2016), an expanded information structure can achieve the BCE as a Bayes Nash equilibrium (BNE). In other words, by providing additional signals to the players, a BCE decision rule can be decentralized as a BNE.

2.1 Single vehicle

We first apply the idea of Bayesian persuasion (Kamenica and Gentzkow, 2011) and consider the following benchmark setting. There is only one vehicle. By Little's (1961) Law,[5] if the vehicle chooses Route 1, the waiting time is $\frac{D_1+\theta}{s_1}$; if the vehicle chooses Route 2, the waiting time is $\frac{D_2}{s_2}$.

		State	
		$\theta=\lambda$	$\theta=0$
Action	Route 1	$\frac{D_1+\lambda}{s_1}$	$\frac{D_1}{s_1}$
	Route 2	$\frac{D_2}{s_2}$	$\frac{D_2}{s_2}$

In this setting, the decision rule $\sigma : \Theta \to \Delta(A)$ specifies the probability of choosing Route 1 $p(\theta)$, conditional on the world's true state $\theta \in \Theta$. This decision can be viewed as a stochastic action recommendation. For simplicity, let $\psi(\lambda) = \psi$. Then, $\psi(0) = 1 - \psi$.

Given the vehicle's prior regarding the traffic and its update of the belief by Bayes' rule, then it will obey the recommendation if the following inequalities hold.

$$\frac{D_1+\lambda}{s_1}p(\lambda)\psi + \frac{D_1}{s_1}p(0)(1-\psi) \le \frac{D_2}{s_2}p(\lambda)\psi + \frac{D_2}{s_2}p(0)(1-\psi), \tag{1}$$

$$\begin{aligned}\frac{D_2}{s_2}(1-p(\lambda))\psi + \frac{D_2}{s_2}(1-p(0))(1-\psi)\\ \le \frac{D_1+\lambda}{s_1}(1-p(0))\psi + \frac{D_1}{s_1}(1-p(0))(1-\psi).\end{aligned} \tag{2}$$

[5] Little's (1961) Law, where $l = \lambda w$, asserts that the time-averaged number of customers in a queueing system, l, is equal to the rate at which customers arrive and enter the system, λ, times the average sojourn time of a customer, w.

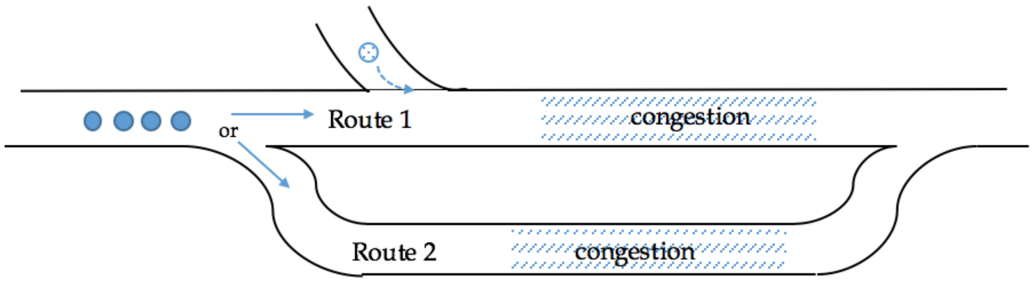

Figure 2: The model of a single vehicle departing at a time

Inequality (1) means that, if Waze asks the vehicle to choose Route 1, then the vehicle has no incentive to deviate from thst recommendation. Note that, $\frac{p(\lambda)\psi}{p(\lambda)\psi+p(0)(1-\psi)}$ is the ex ante probability of $\theta=\lambda$, and $\frac{p(0)(1-\psi)}{p(\lambda)\psi+p(0)(1-\psi)}$ is the ex ante probability of $\theta=0$. Thus, inequality (1) means that, the posterior expectation of the waiting time if driving on Route 1 is less than that of driving on Route 2. Hence, the vehicle follows the recommendation. Similarly, inequality (2) means that the vehicle will choose Route 2 if it is recommended to do so.

The objective of Waze is to minimize the vehicle's expected waiting time of the vehicle. In other words, the Waze's problem (P1) is

$$
\begin{aligned}
\text{(P1)}\quad \inf_{p(\theta)}\quad & \mathbb{E}_\theta\left[\left(\frac{D_1+\theta}{s_1}\right)p(\theta)+\left(\frac{D_2}{s_2}\right)(1-p(\theta))\right] \\
\text{s.t.}\quad & \frac{D_1+\lambda}{s_1}p(\lambda)\psi+\frac{D_1}{s_1}p(0)(1-\psi)\le\frac{D_2}{s_2}p(\lambda)\psi+\frac{D_2}{s_2}p(0)(1-\psi), \\
& \frac{D_2}{s_2}(1-p(\lambda))\psi+\frac{D_2}{s_2}(1-p(0))(1-\psi)\le\frac{D_1+\lambda}{s_1}(1-p(0))\psi+\frac{D_1}{s_1}(1-p(0))(1-\psi), \\
& 0\le p(\theta)\le 1, \forall\theta.
\end{aligned}
$$

Denoting the optimal value of the above problem by $V_1(D_1, D_2)$, we obtain the following proposition that characterizes the optimal solution and the value function.

Proposition 1. *The optimal solution to Waze's problem (P1) is given by*

(i) if $\frac{D_2}{s_2}\le\frac{D_1}{s_1}$*,* $p^*(\lambda)=p^*(0)=0$*,* $V_1(D_1,D_2)=\frac{D_2}{s_2}$*;*

(ii) if $\frac{D_1}{s_1}<\frac{D_2}{s_2}\le\frac{D_1+\lambda}{s_1}$*,* $p^*(\lambda)=0,\ p^*(0)=1$*,* $V_1(D_1,D_2)=\frac{D_1+\lambda}{s_1}(1-\psi)+\frac{D_2}{s_2}\psi$*;*

(iii) if $\frac{D_1+\lambda}{s_1}<\frac{D_2}{s_2}$*,* $p^*(\lambda)=1,\ p^*(0)=1$*,* $V_1(D_1,D_2)=\frac{D_1+\psi\lambda}{s_1}$*.*

Note that, when $\frac{D_2}{s_2} \leq \frac{D_1}{s_1}$ or $\frac{D_1+\lambda}{s_1} < \frac{D_2}{s_2}$, Waze's recommendation is consistent with the vehicle's decentralized decisions. In contrast, when $\frac{D_1}{s_1} < \frac{D_2}{s_2} \leq \frac{D_1+\lambda}{s_1}$, Waze makes different recommendations given different traffic states. Without Waze, the vehicle compares the expected waiting time if choosing Route 1, which is $\frac{D_1+\psi\lambda}{s_1}$, and the waiting time if taking Route 2, $\frac{D_2}{s_2}$. Thus, the waiting time of a vehicle without Waze is given by

$$U_1(D_1, D_2) = \min\{\frac{D_1 + \psi\lambda}{s_1}, \frac{D_2}{s_2}\}.$$

Comparing $V_1(D_1, D_2)$ with $U_1(D_1, D_2)$, we conclude that, in terms of expectations, Waze helps a vehicle to save a waiting time of

$$U_1(D_1, D_2) - V_1(D_1, D_2) = \min\left\{\psi\left(\frac{D_1+\lambda}{s_1} - \frac{D_2}{s_2}\right), (1-\psi)\left(\frac{D_2}{s_2} - \frac{D_1}{s_1}\right)\right\} > 0.$$

2.2 Two vehicles

Consider the case of two vehicles. If both vehicles choose to drive on the same route, no matter the state of the route, both vehicles experience more traffic. Specifically, if both vehicles choose Route 1, then the average waiting time of each vehicle is given by $(D_1 + \theta + \frac{1}{2})/s_1$; if both vehicles choose Route 2, then the average waiting time of each vehicle is given by $(D_2 + \frac{1}{2})/s_2$. Thus, we can summarize the waiting times for different states and strategies in the following tables.

vehicle 2

	$\theta = \lambda$	Route 1	Route 2
vehicle 1	Route 1	$(\frac{D_1+\lambda+\frac{1}{2}}{s_1}, \frac{D_1+\lambda+\frac{1}{2}}{s_1})$	$(\frac{D_1+\lambda}{s_1}, \frac{D_2}{s_2})$
	Route 2	$(\frac{D_2}{s_2}, \frac{D_1+\lambda}{s_1})$	$(\frac{D_2+\frac{1}{2}}{s_2}, \frac{D_2+\frac{1}{2}}{s_2})$

vehicle 2

	$\theta = 0$	Route 1	Route 2
vehicle 1	Route 1	$(\frac{D_1+\frac{1}{2}}{s_1}, \frac{D_1+\frac{1}{2}}{s_1})$	$(\frac{D_1}{s_1}, \frac{D_2}{s_2})$
	Route 2	$(\frac{D_2}{s_2}, \frac{D_1}{s_1})$	$(\frac{D_2+\frac{1}{2}}{s_2}, \frac{D_2+\frac{1}{2}}{s_2})$

In this case, a decision rule consists of the stochastic recommendation for a vehicle to choose Route 1 as in §2.1. It also includes the probability of both vehicles choosing Route 1 under each state. We define $p(n|\theta)$ as the probability of n vehicles choosing Route 1, $n = 0, 1, 2$. Then, the

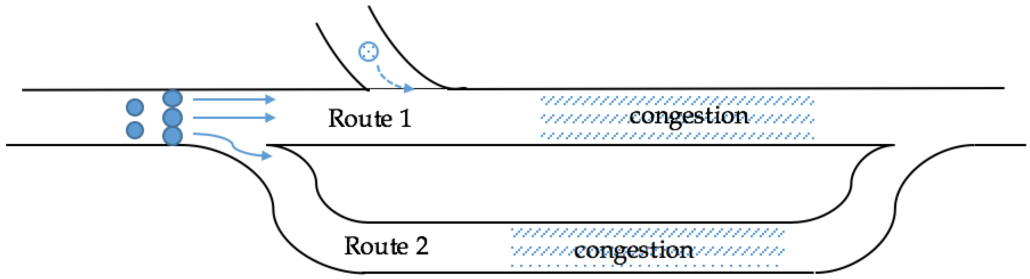

Figure 3: The model of multiple vehicles departing at the same time

obedience conditions for a representative vehicle are given by

$$\mathbb{E}_\theta\left[\frac{D_1+\theta+\frac{1}{2}}{s_1}p(2|\theta)+\frac{D_1+\theta}{s_1}\frac{p(1|\theta)}{2}\right]\leq\mathbb{E}_\theta\left[\frac{D_2}{s_2}p(2|\theta)+\frac{D_2+\frac{1}{2}}{s_2}\frac{p(1|\theta)}{2}\right], \tag{3}$$

$$\mathbb{E}_\theta\left[\frac{D_2+\frac{1}{2}}{s_2}p(0|\theta)+\frac{D_2}{s_2}\frac{p(1|\theta)}{2}\right]\leq\mathbb{E}_\theta\left[\frac{D_1+\theta}{s_1}p(0|\theta)+\frac{D_1+\theta+\frac{1}{2}}{s_1}\frac{p(1|\theta)}{2}\right]. \tag{4}$$

Then, the Waze's problem can be written as

$$\begin{aligned}
\text{(P2)}\quad \inf_{p(n|\theta)}\quad & \mathbb{E}_\theta\left[2\left(\frac{D_1+\theta+\frac{1}{2}}{s_1}\right)p(2|\theta)+\left(\frac{D_1+\theta}{s_1}+\frac{D_2}{s_2}\right)p(1|\theta)+2\left(\frac{D_2+\frac{1}{2}}{s_2}\right)p(0|\theta)\right]\\
\text{s.t.}\quad & (3)\ \&\ (4),\\
& \sum_{n=0}^{2}p(n|\theta)=1,\forall\theta,\\
& 0\leq p(n|\theta)\leq 1,\forall\theta,n=0,1,2.
\end{aligned}$$

Let us denote the optimal value of the above problem by $V_2(D_1,D_2)$. The following proposition characterizes the optimal solution and the value function.

Proposition 2. *The value function of (P2) is given by*

$$\begin{aligned}
V_2(D_1,D_2) = \quad & \sup_{u_1',u_2'\geq 0} \quad \psi \min\left\{ 2\left(\frac{D_1+\lambda+\frac{1}{2}}{s_1}\right) + \left(\frac{D_1+\lambda+\frac{1}{2}}{s_1} - \frac{D_2}{s_2}\right) u_1', \right. \\
& \left(\frac{D_1+\lambda}{s_1} + \frac{D_2}{s_2}\right) + \left(\frac{D_1+\lambda}{2s_1} - \frac{D_2+\frac{1}{2}}{2s_2}\right) u_1' + \left(\frac{D_2}{2s_2} - \frac{D_1+\lambda+\frac{1}{2}}{2s_1}\right) u_2', \\
& \left. 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right) + \left(\frac{D_2+\frac{1}{2}}{s_2} - \frac{D_1+\lambda}{s_1}\right) u_2' \right\} \\
& +(1-\psi)\min\left\{ 2\left(\frac{D_1+\frac{1}{2}}{s_1}\right) + \left(\frac{D_1+\frac{1}{2}}{s_1} - \frac{D_2}{s_2}\right) u_1', \right. \\
& \left(\frac{D_1}{s_1} + \frac{D_2}{s_2}\right) + \left(\frac{D_1}{2s_1} - \frac{D_2+\frac{1}{2}}{2s_2}\right) u_1' + \left(\frac{D_2}{2s_2} - \frac{D_1+\frac{1}{2}}{2s_1}\right) u_2', \\
& \left. 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right) + \left(\frac{D_2+\frac{1}{2}}{s_2} - \frac{D_1}{s_1}\right) u_2' \right\}.
\end{aligned}$$

i) If $\frac{D_2+\frac{1}{2}}{s_2} \leq \frac{D_1}{s_1}$, $p^*(1|\theta) = p^*(2|\theta) = 0,\ p^*(0|\theta) = 1,\ \forall\theta.$

$$V_2(D_1,D_2) = 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right).$$

ii) If $\frac{D_1}{s_1} < \frac{D_2+\frac{1}{2}}{s_2} \leq \frac{D_1+\psi\lambda}{s_1}$,

$$\begin{aligned}
V_2(D_1,D_2) = \quad & \sup_{u_1'\geq 0} \quad \psi \min\left\{ 2\left(\frac{D_1+\lambda+\frac{1}{2}}{s_1}\right) + \left(\frac{D_1+\lambda+\frac{1}{2}}{s_1} - \frac{D_2}{s_2}\right) u_1', \right. \\
& \left. \left(\frac{D_1+\lambda}{s_1} + \frac{D_2}{s_2}\right) + \left(\frac{D_1+\lambda}{2s_1} - \frac{D_2+\frac{1}{2}}{2s_2}\right) u_1', 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right) \right\} \\
& +(1-\psi)\min\left\{ 2\left(\frac{D_1+\frac{1}{2}}{s_1}\right) + \left(\frac{D_1+\frac{1}{2}}{s_1} - \frac{D_2}{s_2}\right) u_1', \right. \\
& \left. \left(\frac{D_1}{s_1} + \frac{D_2}{s_2}\right) + \left(\frac{D_1}{2s_1} - \frac{D_2+\frac{1}{2}}{2s_2}\right) u_1', 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right) \right\}.
\end{aligned}$$

In particular, if $\frac{D_1+\frac{1}{2}}{s_1} > \frac{D_2}{s_2}$, *then* $p^*(1|0) = 1$ *and* $p^*(2|0) = p^*(0|0) = 0.$

iii) If $\frac{D_1+\psi\lambda+\frac{1}{2}}{s_1} < \frac{D_2}{s_2} \leq \frac{D_1+\lambda+\frac{1}{2}}{s_1}$,

$$\begin{aligned} V_2(D_1,D_2) = \quad \sup_{u_2'\geq 0} \quad & \psi \min\left\{2\left(\frac{D_1+\lambda+\frac{1}{2}}{s_1}\right), \left(\frac{D_1+\lambda}{s_1}+\frac{D_2}{s_2}\right)+\left(\frac{D_2}{2s_2}-\frac{D_1+\lambda+\frac{1}{2}}{2s_1}\right)u_2',\right. \\ & \left. 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right)+\left(\frac{D_2+\frac{1}{2}}{s_2}-\frac{D_1+\lambda}{s_1}\right)u_2'\right\} \\ & +(1-\psi)\min\left\{2\left(\frac{D_1+\frac{1}{2}}{s_1}\right), \left(\frac{D_1}{s_1}+\frac{D_2}{s_2}\right)+\left(\frac{D_2}{2s_2}-\frac{D_1+\frac{1}{2}}{2s_1}\right)u_2',\right. \\ & \left. 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right)+\left(\frac{D_2+\frac{1}{2}}{s_2}-\frac{D_1}{s_1}\right)u_2'\right\}. \end{aligned}$$

In particular, if $\frac{D_2+\frac{1}{2}}{s_2} > \frac{D_1+\lambda}{s_1}$, *then* $p^*(1|\lambda)=1$ *and* $p^*(2|\lambda)=p^*(0|\lambda)=0$.

iv) If $\frac{D_2}{s_2} > \frac{D_1+\lambda+\frac{1}{2}}{s_1}$, $p^*(0|\theta)=p^*(1|\theta)=0,\ p^*(2|\theta)=1,\ \forall\theta$.

$$V_2(D_1,D_2) = 2\left(\frac{D_2+\psi\lambda+\frac{1}{2}}{s_1}\right).$$

The above proposition illustrates the solution to the two-vehicle problem. Specifically, when there is rather long queue on Route 1 compared to Route 2, that is $\frac{D_2+\frac{1}{2}}{s_2} \leq \frac{D_1}{s_1}$, Waze routes both departing vehicles to Route 2. In ii) of Proposition 2, $\frac{D_1}{s_1} < \frac{D_2+\frac{1}{2}}{s_2} \leq \frac{D_1+\psi\lambda}{s_1}$. The condition indicates that, the waiting time for both vehicles to go through Route 2 is less than the expected waiting time of either vehicle to go through Route 1, but longer than the waiting time for a single vehicle to choose Route 1 if there is no other traffic on it. Thus, the key is to make a vehicle obey the recommendation of choosing Route 1 if it is optimal to do so. Technically, in this region, we only need to focus on obedience condition (3). Moreover, if $\frac{D_1+\frac{1}{2}}{s_1} > \frac{D_2}{s_2}$, then, when no traffic merging onto Route 1, the optimal solution is to route one vehicle to Route 1 and the other to Route 2. Thus, we will provide personalized private information to both vehicles such that their beliefs about the optimal route are different.

Similarly, when the traffic queue on Route 2 is significantly longer than that on Route 1, it is optimal to route both vehicles to Route 1. In iii) of Proposition 2, the condition $\frac{D_1+\psi\lambda+\frac{1}{2}}{s_1} < \frac{D_2}{s_2} \leq \frac{D_1+\lambda+\frac{1}{2}}{s_1}$ means that, in terms of expectations, both vehicles taking Route 1 is better than a single vehicle taking Route 2, but is worse when traffic is merging onto Route 1. Therefore, it is critical to let vehicles obey the instruction of choosing Route 2 when it is beneficial to do so. In other words, we focus on obedience condition (4). On top of this, if $\frac{D_2+\frac{1}{2}}{s_2} > \frac{D_1+\lambda}{s_1}$, then it is optimal to give different recommendations to the two vehicles.

To further analyze the problem, we first introduce the following theorem.

Theorem 1 (Bergemann and Morris, 2016)**.** *A decision rule σ is a BCE of (G, S) if and only if, for some expansion S^* of S, there is a BNE of (G, S^*) that induces σ.*

The above theorem indicates that a BNE of game (G, S) induces a BCE decision rule. A direct implication is that, without Waze's specific instruction, the outcome of the game – the total waiting time of the two vehicles – is worse. To illustrate, let $U_2(D_1, D_2)$ be the total waiting time of the two vehicles under BNE. Then,

$$\begin{aligned}
&U_2(D_1, D_2) = \\
&\sup_{u_1', u_2' \geq 0} \quad \min\Bigg\{ 2\left(\frac{D_1 + \psi\lambda + \frac{1}{2}}{s_1}\right) + \left(\frac{D_1 + \psi\lambda + \frac{1}{2}}{s_1} - \frac{D_2}{s_2}\right) u_1', \\
&\qquad \left(\frac{D_1 + \psi\lambda}{s_1} + \frac{D_2}{s_2}\right) + \left(\frac{D_1 + \psi\lambda}{2s_1} - \frac{D_2 + \frac{1}{2}}{2s_2}\right) u_1' + \left(\frac{D_2}{2s_2} - \frac{D_1 + \psi\lambda + \frac{1}{2}}{2s_1}\right) u_2', \\
&\qquad 2\left(\frac{D_2 + \frac{1}{2}}{s_2}\right) + \left(\frac{D_2 + \frac{1}{2}}{s_2} - \frac{D_1 + \psi\lambda}{s_1}\right) u_2' \Bigg\}. \qquad (5)
\end{aligned}$$

It appears that $U_2(D_1, D_2) \geq V_2(D_1, D_2)$. Note that, Waze is particularly useful when $U_2(D_1, D_2) > V_2(D_1, D_2)$, which indicates that, with the vehicles' following Waze's instructions, the total waiting time is strictly decreased. This situation could arise when $\frac{D_1}{s_1} - \frac{1}{2s_2} < \frac{D_2}{s_2} < \frac{D_1+\lambda+\frac{1}{2}}{s_1}$. In other words, when there is no huge difference in waiting time between two routes and the routing decisions vary with the traffic's exact situation, the augmented reality generated by Waze will significantly adjust the vision of the vehicles.

Now, we characterize the optimal information structure Waze should provide. We focus on the case in which $\frac{D_1}{s_1} < \frac{D_2+\frac{1}{2}}{s_2} \leq \frac{D_1+\psi\lambda}{s_1}$ and $\frac{D_1+\frac{1}{2}}{s_1} > \frac{D_2}{s_2}$, where the optimal value function is shown in ii) of Proposition 2. Consider the following three information structures: (1) no augmented reality, (2) complete information, and (3) an optimal BCE information structure.

1. No augmented reality

If no additional information is provided to the vehicles, then the outcome is a BNE and the resultant total waiting time is given by $U_2(D_1, D_2)$. From (5), we obtain

$$U_2(D_1, D_2) = 2\left(\frac{D_2 + \frac{1}{2}}{s_2}\right).$$

Under the BNE, both vehicles choose Route 2.

2. Complete information

Waze lets both vehicles know the exact state of the traffic. If $\theta = \lambda$, both vehicles choose Route 2. If $\theta = 0$, then there are three equilibria: (Route 1, Route 2), (Route 2, Route 1), and the mixed (p_mRoute 1+$(1-p_m)$ Route 2, p_mRoute 1+$(1-p_m)$Route 2) with $p_m = 2\left(\frac{D_2}{s_2} - \frac{D_1}{s_1}\right)/(\frac{1}{s_1} + \frac{1}{s_2})$.[6] Thus, the expected total waiting time is

$$\begin{aligned} U_2^C(D_1, D_2) &= 2\psi\left(\frac{D_2 + \frac{1}{2}}{s_2}\right) + (1-\psi)\left[p_m\left(\frac{D_1 + \frac{1}{2}}{s_1} + \frac{D_2}{s_2}\right) + (1-p_m)\left(\frac{D_1}{s_1} + \frac{D_2 + \frac{1}{2}}{s_2}\right)\right] \\ &\leq U_2(D_1, D_2). \end{aligned}$$

3. Optimal BCE information structure

Consider two possible signals $T_i = \{t_f, t_s\}$ given to vehicle i and a distribution $\pi : \Theta \to \Delta(T)$ where $T = T_1 \times T_2$. If $\theta = \lambda$, then both vehicles receive the same information indicating that the traffic is relatively light on Route 2. If $\theta = 0$, then either vehicle (but not both) receives the information indicating that the traffic on Route 1 is relative light. Formally, the distribution π is

$\pi(\cdot\|0)$	t_f	t_s
t_f	0	0
t_s	0	1

$\pi(\cdot\|\lambda)$	t_f	t_s
t_f	0	1/2
t_s	1/2	0

Provided with the information structure (T, π), vehicle i has the BNE strategy

$$\beta_i(t_i) = \begin{cases} \text{Route 1,} & \text{if } t_i = t_f, \\ \text{Route 2} & \text{if } t_i = t_s. \end{cases}$$

Under this strategy, the total waiting time of the wo vehicles is

$$\begin{aligned} V_2(D_1, D_2) &= \psi\left(\frac{D_2 + \frac{1}{2}}{s_2}\right) + (1-\psi)\left(\frac{D_1}{s_1} + \frac{D_2}{s_2}\right) \\ &\leq U_2^C(D_1, D_2). \end{aligned}$$

In summary, $V_2(D_1, D_2) \leq U_2^C(D_1, D_2) \leq U_2(D_1, D_2)$. That is, among the three information structures we present, providing complete information is better than providing no information, and the optimal BCE information structure leads to the shortest total waiting time.

[6] We assume $\frac{D_2}{s_2} > \frac{D_1}{s_1}$. Other cases can be analyzed similarly.

It is interesting that providing the same additional accurate information to two vehicles may not be the best strategy. In fact, to coordinate vehicles, a better strategy is to privately communicate different information when splitting traffic flow will induce socially efficient use of the road system.

3 Dynamic models

3.1 Single vehicle departing at given times

Consider a periodic review model of traffic routing (Figure 2). Two parallel routes connect the starting point and the destination: Route 1 and Route 2. The capacity of a bottleneck on route j is assumed to be s_j. At the beginning of each period, one vehicle departs from the starting point. The vehicle is informed about D_j, the number of vehicles in the queue on route j, by the end of the last period. We assume other vehicles can merge onto Route 1 and join the queue. Thus, if the vehicle chooses Route 1, by the time it joins the queue, the queue length will be $D_1 + \theta$, where

$$\theta = \begin{cases} \lambda, & \text{with probability } \psi, \\ 0, & \text{with probability } 1-\psi. \end{cases}$$

The realization of θ is known only by Waze and not by the vehicle. On the other hand, if the vehicle chooses Route 2, by the time it joins the queue, the queue length will be D_2.

When the vehicle departs, Waze makes a recommendation $p(\theta)$ to the vehicle based on the realization of θ. Let $p(0)$ and $p(\lambda)$ be the stochastic recommendations of choosing Route 1 given $\theta = 0$ and $\theta = \lambda$, respectively. Then, the vehicle will follow the recommendation if

$$\frac{D_1+\lambda}{s_1}p(\lambda)\psi + \frac{D_1}{s_1}p(0)(1-\psi) \le \frac{D_2}{s_2}p(\lambda)\psi + \frac{D_2}{s_2}p(0)(1-\psi), \tag{6}$$

$$\begin{aligned} &\frac{D_2}{s_2}(1-p(\lambda))\psi + \frac{D_2}{s_2}(1-p(0))(1-\psi) \\ &\le \frac{D_1+\lambda}{s_1}(1-p(0))\psi + \frac{D_1}{s_1}(1-p(0))(1-\psi). \end{aligned} \tag{7}$$

Waze's problem can be formulated as a dynamic programming problem. Let $V_3(D_1, D_2)$ be the profit-to-go function starting from the state (D_1, D_2). The Bellman equation is given by

$$
\begin{aligned}
& V_3(D_1,D_2) \quad = \\
\text{(P3)} \quad & \inf_{p(\theta)} \quad \mathbb{E}_\theta \left\{ \left(\frac{D_1+\theta}{s_1} + V_3((D_1+\theta+1-s_1)^+, (D_2-s_2)^+) \right) p(\theta) \right. \\
& \qquad \left. + \left(\frac{D_2}{s_2} + V_3((D_1+\theta-s_1)^+, (D_2+1-s_2)^+) \right) (1-p(\theta)) \right\} \\
& \text{s.t.} \quad \frac{D_1+\lambda}{s_1} p(\lambda)\psi + \frac{D_1}{s_1} p(0)(1-\psi) \le \frac{D_2}{s_2} p(\lambda)\psi + \frac{D_2}{s_2} p(0)(1-\psi), \\
& \qquad \frac{D_2}{s_2}(1-p(\lambda))\psi + \frac{D_2}{s_2}(1-p(0))(1-\psi) \\
& \qquad \le \frac{D_1+\lambda}{s_1}(1-p(\lambda))\psi + \frac{D_1}{s_1}(1-p(0))(1-\psi), \\
& \qquad 0 \le p(\theta) \le 1, \forall \theta.
\end{aligned}
$$

The terminal condition is given by $V_3(0,0) = 0$.

First, note that, when $D_1 \le s_1$ and $D_2 \le s_2$, the above problem is equivalent to (P1). Therefore,

$$V_3(D_1,D_2) = V_1(D_1,D_2), \quad \text{if } D_1 \le s_1 \text{ and } D_2 \le s_2.$$

Next, we separate the state space into four regions.

Region I. $\frac{D_2}{s_2} \le \frac{D_1}{s_1}$.

In this region, there is no strictly positive $p(\theta)$ such that constraint (6) is satisfied. Thus, $p(\theta) = 0, \forall \theta$. This means that Waze is unable to change the choice of the vehicle and the vehicle will choose Route 2. Thus, we obtain

$$V_3(D_1,D_2) = \frac{D_2}{s_2} + \mathbb{E}_\theta V_3((D_1+\theta-s_1)^+, (D_2+1-s_2)^+).$$

Region II. $\frac{D_1}{s_1} < \frac{D_2}{s_2} \le \frac{D_1+\psi\lambda}{s_1}$.

In this region, constraint (6) implies constraint (7). Thus, we can ignore constraint (7).

Region III. $\frac{D_1+\psi\lambda}{s_1} < \frac{D_2}{s_2} \le \frac{D_1+\lambda}{s_1}$.

In this region, constraint (7) implies constraint (6). Thus, we can ignore constraint (6).

Region IV. $\frac{D_1+\lambda}{s_1} < \frac{D_2}{s_2}$.

In this region, there is no $p(\theta)$ strictly less than one such that constraint (7) is satisfied. Thus, $p(\theta) = 1, \forall \theta$. This means that Waze is unable to change the choice of the vehicle and the vehicle will choose Route 1.

Next, we demonstrate the improvement in efficiency achieved by Waze. Let $U_3(D_1, D_2)$ be the total expected waiting time of the vehicles without Waze. Each departing vehicle chooses the route with the shortest expected waiting time. Then,

$$U_3(D_1, D_2) \quad = \quad \begin{cases} \frac{D_1+\psi\lambda}{s_1} + \mathbb{E}_\theta U_3((D_1+1+\theta-s_1)^+, (D_2-s_2)^+), & \text{if } \frac{D_1+\psi\lambda}{s_1} \le \frac{D_2}{s_2}, \\ \frac{D_2}{s_2} + \mathbb{E}_\theta U_3((D_1+\theta-s_1)^+, (D_2+1-s_2)^+), & \text{if } \frac{D_1+\psi\lambda}{s_1} > \frac{D_2}{s_2}. \end{cases}$$

To see that $U_3(D_1, D_2) \ge V_3(D_1, D_2)$, we rewrite $U_3(D_1, D_2)$ as the value function for the following programming (P3').

$$\begin{aligned}
& U_3(D_1, D_2) \\
= \quad & \inf_p \quad \mathbb{E}_\theta \left\{ \left(\frac{D_1+\psi\lambda}{s_1} + V((D_1+\theta+1-s_1)^+, (D_2-s_2)^+) \right) p \right. \\
& \qquad \left. + \left(\frac{D_2}{s_2} + V((D_1+\theta-s_1)^+, (D_2+1-s_2)^+) \right) (1-p) \right\} \\
& \text{s.t.} \quad \frac{D_1+\psi\lambda}{s_1} p \le \frac{D_2}{s_2} p, \ \frac{D_2}{s_2}(1-p) \le \frac{D_1+\psi\lambda}{s_1}(1-p), \\
& \qquad p \in \{0, 1\} \\
= \quad & \inf_{p(\theta)} \quad \mathbb{E}_\theta \left\{ \left(\frac{D_1+\theta}{s_1} + V((D_1+\theta+1-s_1)^+, (D_2-s_2)^+) \right) p(\theta) \right. \\
& \qquad \left. + \left(\frac{D_2}{s_2} + V((D_1+\theta-s_1)^+, (D_2+1-s_2)^+) \right) (1-p(\theta)) \right\} \\
& \text{s.t.} \quad \frac{D_1+\lambda}{s_1} p(\lambda)\psi + \frac{D_1}{s_1} p(0)(1-\psi) \le \frac{D_2}{s_2} p(\lambda)\psi + \frac{D_2}{s_2} p(0)(1-\psi), \\
& \qquad \frac{D_2}{s_2}(1-p(\lambda))\psi + \frac{D_2}{s_2}(1-p(0))(1-\psi) \\
& \qquad \le \frac{D_1+\lambda}{s_1}(1-p(\lambda))\psi + \frac{D_1}{s_1}(1-p(0))(1-\psi), \\
& \qquad p(0) = p(\lambda), \\
& \qquad 0 \le p(\theta) \le 1, \forall \theta.
\end{aligned}$$

From the above transformation, we can see that the feasible set of (P3') is a subset of the that of (P3). In addition, programming models (P3) and (P3') have the same objective function. Thus, we obtain $U_3(D_1, D_2) \ge V_3(D_1, D_2)$. Since the total waiting time is reduced under Waze's instruction, we conclude that the innovative Waze makes the traffic system more efficient.

3.2 Multiple vehicles departing at given times

We study a dynamic traffic congestion model over a time period $[0, T]$ (Figure 3). In this general model, we formulate a discrete time model with time intervals of length h. At the beginning of each time interval, Waze counts the number of vehicles that are ready to depart. At time t, $I(t)$ vehicles are departing.[7] We again consider two parallel routes, Route 1 and Route 2. Each route j has two segments with fixed travel times, h and t_0^j, repectively. A delay can occur on each route at a bottleneck on the second segment. The capacity of a bottleneck on route j is assumed to be s_j. That is, after driving for a time h, a vehicle might see the traffic queue. If there is congestion on a route, then the travel time of a vehicle with departure time t is the sum of fixed travel times and the delay time $t_\nu^j(t|\theta)$. In other words, the total travel time of a vehicle departing at t on route j is

$$tt^j(t|\theta) = h + t_0^j + t_\nu^j(t|\theta).$$

Let $D_j(t|\theta)$ be the number of vehicles in the queue on route j at time t. By Little's Law, we obtain

$$t_\nu^j(t|\theta) = \frac{D_j(t+h|\theta)}{s_j}.$$

Let $R_j(t)$ be the number of vehicles departing in the time interval $[t, t+h]$ on route j. Then, the dynamics of $D_j(t|\theta)$ are

$$\begin{aligned} D_j(t+h|\theta) &= \begin{cases} D_j(t|\theta) + \frac{1}{2}(R_j(t)-1)^+ - hs_j & \text{for congestion,} \\ 0 & \text{for no congestion,} \end{cases} \\ &= [D_j(t|\theta) + \frac{1}{2}(R_j(t)-1)^+ - hs_j]^+. \end{aligned}$$

Thus,

$$tt^j(t|\theta) = h + t_0^j + \frac{D_j(t|\theta) + \frac{1}{2}(R_j(t)-1)^+ - hs_j}{s_j} \quad \text{for congestion.}$$

The state of the world θ specifies the queue length on each route at time $t = 0$.

Now we consider the vehicle's strategy. Vehicle i departing in the time interval $[t, t+h]$ can

[7] We assume h is given. In practice, it would be interesting to study how to set the length of a time interval. The tradeoff is that, if h is small, then the problem size is small. However, Waze has to compute the optimal solution quickly. If h is set to be long, then the problem size is large, but the speed requirement for computation is lower.

choose either Route 1 or Route 2. Let $a_i(t)$ be the vehicle's strategy,

$$a_i(t) = \begin{cases} 1 & \text{if vehicle } i \text{ chooses Route 1,} \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} R_1(t) &= \sum_{i=1}^{I(t)} a_i(t), \\ R_2(t) &= \sum_{i=1}^{I(t)} (1 - a_i(t)) = I(t) - \sum_{i=1}^{I(t)} a_i(t). \end{aligned}$$

For simplicity, the utility function is defined as the negative of the total travel time.

$$\begin{aligned} u_1((a_i(t), a_{-i}(t))|\theta) &= -\left[h + t_0^1 + \frac{D_1(t|\theta) + \frac{1}{2}(\sum_{i=1}^{I(t)} a_i(t) - 1)^+ - hs_1}{s_1} \right], \\ u_2((a_i(t), a_{-i}(t))|\theta) &= -\left[h + t_0^2 + \frac{D_2(t|\theta) + \frac{1}{2}(\sum_{i=1}^{I(t)} (1 - a_i(t)) - 1)^+ - hs_2}{s_2} \right]. \end{aligned}$$

Similar to previous sections, the decision rule $\sigma_t : \Theta \to \Delta(A)$ specifies the probability over the action set A conditional on the world's true state at time t. Note that, here, we focus on symmetric decision rules, given that the basic game is symmetric and we will consider a symmetric objective function. The obedience condition is given by the linear inequality:

$$\sum_{a_{-i} \in A_{-i}, \theta \in \Theta} u_1((a_i = 1, a_{-i})|\theta) \sigma_t((a_i = 1, a_{-i})|\theta) \geq \sum_{a_{-i} \in A_{-i}, \theta \in \Theta} u_2((a_i = 0, a_{-i})|\theta) \sigma_t((a_i = 1, a_{-i})|\theta),$$

and

$$\sum_{a_{-i} \in A_{-i}, \theta \in \Theta} u_2((a_i = 0, a_{-i})|\theta) \sigma_t((a_i = 0, a_{-i})|\theta) \geq \sum_{a_{-i} \in A_{-i}, \theta \in \Theta} u_1((a_i = 1, a_{-i})|\theta) \sigma_t((a_i = 0, a_{-i})|\theta).$$

The objective function is defined as

$$\text{(P4)} \quad \max \mathbb{E}_\theta \left[\sum_t \sum_{\mathbf{a}(t) \in A} \left(\sum_i a_i(t) u_1(\mathbf{a}(t)|\theta) + \sum_i (1 - a_i(t)) u_2(\mathbf{a}(t)|\theta) \right) \sigma_t(\mathbf{a}(t)|\theta) \right].$$

Note that this optimization problem has $\mathcal{O}(2^{I(t)})$ decision variables for period t. However, by the symmetry of the game, we can reduce the number of decision variables to $\mathcal{O}(I(t))$ for period t. Specifically, we can write

$$\sum_{\sum a_i(t)=n} \sigma_t(\mathbf{a}(t))|\theta) = p(n,t|\theta), \quad n = 0, \cdots, I(t),$$

where $p(n,t|\theta)$ is the probability of n vehicles choosing Route 1 at time t. Then the obedience condition can be written as

$$\begin{aligned}
& \sum_{n\in\{1,\cdots,I(t)\},\theta\in\Theta} -\left[h + t_0^1 + \frac{D_1(t|\theta) + \frac{1}{2}(n-1) - hs_1}{s_1}\right] \frac{n}{I(t)} p(n,t|\theta)\psi(\theta) \\
\geq \quad & \sum_{n\in\{1,\cdots,I(t)\},\theta\in\Theta} -\left[h + t_0^2 + \frac{D_2(t|\theta) + \frac{1}{2}(I(t)-n) - hs_2}{s_2}\right] \frac{n}{I(t)} p(n,t|\theta)\psi(\theta) \quad \forall t,
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{n\in\{0,\cdots,I(t)-1\},\theta\in\Theta} -\left[h + t_0^2 + \frac{D_2(t|\theta) + \frac{1}{2}(I(t)-n-1) - hs_2}{s_2}\right] \frac{I(t)-n}{I(t)} p(n,t|\theta)\psi(\theta) \\
\geq \quad & \sum_{n\in\{0,\cdots,I(t)-1\},\theta\in\Theta} -\left[h + t_0^1 + \frac{D_1(t|\theta) + \frac{1}{2}n - hs_1}{s_1}\right] \frac{I(t)-n}{I(t)} p(n,t|\theta)\psi(\theta) \quad \forall t.
\end{aligned}$$

The objective function is transformed into

$$\begin{aligned}
\min \mathbb{E}_\theta \Bigg\{ \sum_{n\in\{0,\cdots,I(t)\},t} \Bigg[& \left(t_0^1 + \frac{D_1(t|\theta) + \frac{1}{2}(n-1)}{s_1}\right) n \\
& + \left(t_0^2 + \frac{D_2(t|\theta) + \frac{1}{2}(I(t)-n-1)}{s_2}\right)(I(t)-n) \Bigg] p(n,t|\theta) \Bigg\}.
\end{aligned}$$

In sum, we have the following linear programming (P_{BCE}):

$$\begin{aligned}
\min \quad & \sum_{\substack{n\in\{0,\cdots,I(t)\}\\ t=1,\cdots,T,\theta\in\Theta}} \Bigg[\left(t_0^1+\frac{D_1(t|\theta)+\frac{1}{2}(n-1)}{s_1}\right)n \\
& +\left(t_0^2+\frac{D_2(t|\theta)+\frac{1}{2}(I(t)-n-1)}{s_2}\right)(I(t)-n)\Bigg]p(n,t|\theta)\psi(\theta) \\
\text{s.t.} \quad & \sum_{n\in\{1,\cdots,I(t)\},\theta\in\Theta} -\left[h+t_0^1+\frac{D_1(t|\theta)+\frac{1}{2}(n-1)-hs_1}{s_1}\right]\frac{n}{I(t)}p(n,t|\theta)\psi(\theta) \\
& \geq \sum_{n\in\{1,\cdots,I(t)\},\theta\in\Theta} -\left[h+t_0^2+\frac{D_2(t|\theta)+\frac{1}{2}(I(t)-n)-hs_2}{s_2}\right]\frac{n}{I(t)}p(n,t|\theta)\psi(\theta) \quad \forall t, \qquad (8) \\
& \sum_{n\in\{0,\cdots,I(t)-1\},\theta\in\Theta} -\left[h+t_0^2+\frac{D_2(t|\theta)+\frac{1}{2}(I(t)-n-1)-hs_2}{s_2}\right]\frac{I(t)-n}{I(t)}p(n,t|\theta)\psi(\theta) \\
& \geq \sum_{n\in\{0,\cdots,I(t)-1\},\theta\in\Theta} -\left[h+t_0^1+\frac{D_1(t|\theta)+\frac{1}{2}n-hs_1}{s_1}\right]\frac{I(t)-n}{I(t)}p(n,t|\theta)\psi(\theta) \quad \forall t, \qquad (9) \\
& \sum_{n\in\{0,\cdots,I(t)\}} p(n,t|\theta)=1 \quad \forall t,\theta, \qquad (10) \\
& p(n,t|\theta)\geq 0 \quad \forall n,t,\theta. \qquad (11)
\end{aligned}$$

Specifically, let $\boldsymbol{\beta}_t=(\beta_{t,1}(a_1(t)),\cdots,\beta_{t,I(t)}(a_{I(t)}(t)))$ be a BNE strategy profile. Then, the BCE strategy $\tilde{\sigma}$ induced by this BNE strategy is defined as

$$\tilde{\sigma}(\mathbf{a}(t))|\theta)\triangleq\prod_{i=1}^{I(t)}\beta_{t,i}(a_i(t)), \quad \forall t,\theta.$$

Let us denote

$$\sum_{\sum a_i(t)=n}\tilde{\sigma}_t(\mathbf{a}(t))|\theta)=\tilde{p}(n,t|\theta), \quad n=0,\cdots,I(t).$$

By Theorem 1, we know that $\tilde{p}$ satisfies conditions (8)–(11). Thus, $\tilde{p}$ is a feasible solution to the following programming (P_{BNE}):

$$
\begin{aligned}
\min \quad & \sum_{\substack{n\in\{0,\cdots,I(t)\}\\ t=1,\cdots,T,\theta\in\Theta}} \left[\left(t_0^1+\frac{D_1(t|\theta)+\frac{1}{2}(n-1)}{s_1}\right)n\right.\\
& \quad \left.+\left(t_0^2+\frac{D_2(t|\theta)+\frac{1}{2}(I(t)-n-1)}{s_2}\right)(I(t)-n)\right]p(n,t|\theta)\psi(\theta)\\
\text{s.t.} \quad & \sum_{n\in\{1,\cdots,I(t)\},\theta\in\Theta} -\left[h+t_0^1+\frac{D_1(t|\theta)+\frac{1}{2}(n-1)-hs_1}{s_1}\right]\frac{n}{I(t)}p(n,t|\theta)\psi(\theta)\\
& \geq \sum_{n\in\{1,\cdots,I(t)\},\theta\in\Theta} -\left[h+t_0^2+\frac{D_2(t|\theta)+\frac{1}{2}(I(t)-n)-hs_2}{s_2}\right]\frac{n}{I(t)}p(n,t|\theta)\psi(\theta) \quad \forall t,\\
& \sum_{n\in\{0,\cdots,I(t)-1\},\theta\in\Theta} -\left[h+t_0^2+\frac{D_2(t|\theta)+\frac{1}{2}(I(t)-n-1)-hs_2}{s_2}\right]\frac{I(t)-n}{I(t)}p(n,t|\theta)\psi(\theta)\\
& \geq \sum_{n\in\{0,\cdots,I(t)-1\},\theta\in\Theta} -\left[h+t_0^1+\frac{D_1(t|\theta)+\frac{1}{2}n-hs_1}{s_1}\right]\frac{I(t)-n}{I(t)}p(n,t|\theta)\psi(\theta) \quad \forall t,\\
& \sum_{n\in\{0,\cdots,I(t)\}} p(n,t|\theta)=1 \quad \forall t,\theta,\\
& p(n,t|\theta)=p(n,t|\theta'), \quad \forall\theta,\theta'\in\Theta, \qquad (12)\\
& p(n,t|\theta)\geq 0 \quad \forall n,t,\theta.
\end{aligned}
$$

Note that the difference between (P_{BCE}) and (P_{BNE}) is that we impose the additional constraint (12) in (P_{BNE}). In fact, all feasible solutions to (P_{BNE}) can define a BNE. Under constraint (12), constraint (8) requires that Route 1 indeed be the best response for each vehicle whose BNE strategy is to choose Route 1; constraint (9) requires that Route 2 indeed be the best response for each vehicle whose BNE strategy is to choose Route 2. Clearly, the optimal value of (P_{BCE}) is less than that of (P_{BNE}). Thus, without Waze, the total travel time of vehicles under a BNE is longer than under a BCE.

We now consider the additional information structure required by the optimal decision rules. Let $p^*(n,t|\theta)$ be the optimal solutions to LP (P_{BCE}). Then, the optimal decision rule is

$$\sigma_t^*(\mathbf{a}(t))|\theta)=\frac{1}{\binom{I(t)}{n}}p^*(n,t|\theta), \quad \forall\mathbf{a}(t):\sum a_i(t)=n, n=0,\cdots,I(t).$$

We consider two signals $T_i=\{t_f,t_s\}$ that might be given to vehicle i and a distribution $\pi:\Theta\rightarrow$

$\Delta(T)$, where $T = T_1 \times \cdots T_{I(t)}$. π satisfies

$$\pi(t_1, \cdots, t_{I(t)}|\theta) = \sigma_t^*(\mathbf{a}(t))|\theta), \text{where } a_i(t) = \begin{cases} 1, & \text{if } t_i = t_f \\ 0 & \text{if } t_i = t_s. \end{cases}$$

Given the information structure $S = (T, \pi)$, a BNE strategy for the basic game (G, S) is

$$\beta_i^*(a_i(t)|t_i) = \begin{cases} 1, & \text{if } t_i = t_f \\ 0 & \text{if } t_i = t_s. \end{cases}$$

The total travel time of the decentralized game (G, S) under the strategy profile β^* will be the optimal value of (P_{BCE}).

4 Urban traffic

In this section, we consider the information design problem for urban traffic (Figure 4). Instead of selecting from two routes, as discussed in previous sections, we consider the routing problem in a urban road network denoted by a directed graph (V, E). Here, V is the set of nodes (intersections) and two special nodes, the source (starting point) s and the sink (destination) t, and $E = (e)$ is the set of edges. Vehicles departing from the source follow a point process (Daley and Vere-Jones, 2007). Each vehicle, upon departure, chooses a path to the sink. Let P be the set of all paths from the source to the sink.

A vehicle departing at time t is informed about the current traffic volume (queue length) on each edge e, $D_e(t)$. However, it is not sure about the queue length at time $t' > t$. Thus, for a path $p = (e_0, \cdots, e_n) \in P$, the vehicle assumes that the queue length is $D_{e_i}(t) + \theta_{e_i}$ when it arrives at the edge e_i, $i = 1, \cdots, n$. $\theta_p = (\theta_{e_0}, \cdots, \theta_{e_n}) \sim F_p(\cdot)$ and $\theta = (\theta_p)_{p \in P} \sim F(\cdot)$. Specifically, from the vehicle's perspective, the travel time along e_0 is given by

$$\tau_{e_0} = \frac{D_{e_0}(t) + 1}{s_{e_0}}.$$

Thus, at time $t + \tau_{e_0}$, the vehicle arrives at edge e_1. However, since it is not sure about $D_{e_1}(t + \tau_{e_0})$ when it departs from s, it assumes $D_{e_1}(t + \tau_{e_0}) = D_{e_1}(t) + \theta_{e_1}$. Its travel time on e_1 is then

$$\tau_{e_1} = \frac{D_{e_1}(t) + \theta_{e_1} + 1}{s_{e_1}}.$$

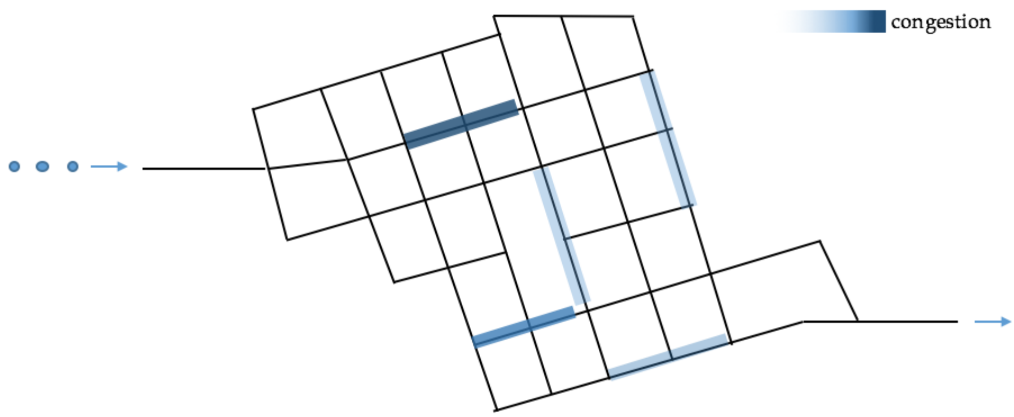

Figure 4: A model of urban traffic

At time $t + \tau_{e_0} + \tau_{e_1}$, the vechile arrives at edge e_2 and so on and so forth. Thus, to the vehicle, the travel time on edge e_i, $\tau_{e_i}(t)$, and the total travel time on path p, $\tau_p(t)$, are

$$\tau_{e_i}(t) = \frac{D_{e_i}(t) + \theta_{e_i} + 1}{s_{e_i}}, \quad \text{and} \quad \tau_p(t) = \sum_{i=0}^{n} \tau_{e_i}(t).$$

Unlike the vehicle, Waze knows the dynamics of $D_e(t)$, which are governed by

$$\frac{\partial D_e(t)}{\partial t} = \begin{cases} r_e(t) - s_e & \text{for congestion,} \\ 0 & \text{for no congestion,} \end{cases}$$

where $r_e(t)$ is the arrival rate at edge e.

The true travel time of a vehicle departing at time t driving on path $p = (e_0, \cdots, e_n) \in P$ is

$$tt_p(t) = t_{e_0}(t) + \cdots + t_{e_n}(t),$$

where

$$\begin{aligned}
t_{e_0}(t) &= \frac{D_{e_0}(t)+1}{s_{e_0}}, \\
t_{e_1}(t) &= \frac{D_{e_1}(t+t_{e_0})+1}{s_{e_1}}, \\
t_{e_2}(t) &= \frac{D_{e_2}(t+t_{e_0}+t_{e_1})+1}{s_{e_2}}, \\
& \dots \\
t_{e_n}(t) &= \frac{D_{e_2}(t+\sum_{i=0}^{n-1} t_{e_i})+1}{s_{e_n}}.
\end{aligned}$$

Waze's problem is to persuade each departing vehicle to choose the socially optimal path through a stochastic path recommendation $\sigma(p|\theta)$ that specifies the probability of selecting path p given θ, subject to the obedience conditions:

$$\mathbb{E}_\theta[\tau_p(t)\sigma(p|\theta)] \le \mathbb{E}_\theta[\tau_{p'}(t)\sigma(p|\theta)], \quad \forall p' \in P.$$

If the vehicle follows path p, then Waze updates its information on the arrival rate at each edge e_i that belongs to the chosen path. Specifically, the updated arrival rate $r'_{e_i}(t')$ for time $t' > t$ is

$$r'_{e_i}(t') = \begin{cases} r_e(t')+1 & \text{if } t' = \sum_{i=0}^{i-1} t_{e_i}(t), \\ r_e(t') & \text{otherwise.} \end{cases} \quad \forall i = 1, \cdots, n.$$

For an edge that is not on the path p, the arrival rate does not change. $r'_e(t') = r_e(t), \forall e \notin p$.

In sum, we can write Waze's information design problem in a urban area as follows:

$$\begin{aligned}
V(r_e(t), e \in E) = \quad & \inf \quad \mathbb{E}_{t'-t}\left[\sum_{p\in P} tt_p(t)\sigma(p|\theta) + V(r'_e(t'), e \in E)\right] \\
& s.t. \quad \mathbb{E}_\theta[\tau_p(t)\sigma(p|\theta)] \le \mathbb{E}_\theta[\tau_{p'}(t)\sigma(p|\theta)], \quad \forall p' \in P. \\
& \qquad r'_{e_i}(t') = \begin{cases} r_e(t')+1 & \text{if } t' = \sum_{i=0}^{i-1} t_{e_i}(t) \\ r_e(t') & \text{otherwise.} \end{cases} \quad \forall i = 1, \cdots, n. \\
& \qquad \sum_{p\in P} \sigma(p|\theta) = 1, \forall \theta, \\
& \qquad 0 \le \sigma(p|\theta) \le 1, \forall p, \theta.
\end{aligned}$$

5 Discussion and Conclusion

In this paper, we propose an economic approach that exploits the traffic data crowdsourced by Waze and designs the information structure provided to autonomous vehicles to minimize the total travel delays. By implementing the optimal solutions suggested by the correlated equilibrium, Waze is able to manage multiple vehicles moving on designated paths, which leads to better overall traffic performance and utilization of the public roads over a long period. Our method of improving the efficiency of the traffic system deviates from the traditional economic approach, in which mechanisms and policy (e.g., congestion pricing and surge pricing) are used to encourage more the effective use of the services and the roads by shifting demand and supply.

We believe that our work is just a first step to theoretically characterizing and optimizing routing problems for autonomous vehicles. Many critical and practical issues remain for further research. We list a few below.

5.1 Heterogeneous vehicles

In this paper, we assume that all vehicles aim to minimize their own waiting times. That is, vehicles have the same tolerance towards traffic delays and we cover the symmetric case of treating all vehicles equally. It could be interesting to study an extension that relaxes this assumption and uses utility functions to describe different characteristics of vehicles including the tightness of schedules and priority.

does maximand matter? what if want to minimize left turns or chances of accident subject to arriving w/in some period of time?

Travelers on a public road system may have different attitudes towards congestion and thus have different degrees of disutility for travel delays. Specifically, some business travelers experience a high disutility from long travel time, since they have tight schedules, whereas leisure travelers are less stressed by traffic jams. Alternatively, in a purely robotic system, a robot might be wanted at a particular location and a given moment while another robot need only arrive at a certain place within a few hours. When designing paths for robots in such a setting, it is critical to take into account the disparity in the costs of delay incurred by different robots. Similarly to modeling human beings, to formally model this disparity, we could also use a "disutility" measure to characterize a robot's level of suffering from the delay. Specifically, we introduce a utility function associated with the travel delay for each vehicle (robot). In a linear representation, the utility of a vehicle i which waits for τ units of time is given by $u_i(\tau) = b - c_i\tau$, where b denotes the utility of arriving at the destination from the starting point and c_i denotes the disutility of waiting per unit of time of vehicle i. Notice that, we allow different vehicles to be associated with different parameter c_i, which characterizes the disparity in tolerance towards delays. In these settings, models become related to that of Kolotilin et al. (n.d.), which is a variation of that of Kamenica and Gentzkow (2011).

Vehicles which suffer more from congestion are more willing to move along the express lane (if there is one) compared to those suffering less from congestion. Thus, to differentiate various types of vehicles, we can then introduce a pricing mechanism besides the information-based approach. Hence, we can study the information/mechanism design problem in a setting where an express toll lane is built for urgent vehicles. Vehicles can indicate their priority preferences and the willingness-to-pay to move to the express lane by paying the toll.

To solve for the optimal toll and the information structure, Waze maximizes the sum of utilities of all vehicles given the traffic conditions. Waze's optimization problem is conditional on obedience conditions as in this paper. Besides, the optimal solution should also satisfy the incentive compatibility constraints and individual rationality constraints as in general mechanism design models.

5.2 Accidents and vehicles with priority

In the real world, traffic accidents usually cause severe congestion and require emergency vehicles such as ambulances and police cars to arrive at the scene as soon as possible to save the injured person, handle the accident, and direct the traffic. Thus, it is important for Waze to react quickly to accidents and route cars to make way for emergency vehicles.

5.3 Intersections

Currently, traffic lights are used to control flows of traffic at road intersections. As autonomous vehicles become pervasive, it will be possible to manage traffic at crossings through Waze. Fundamentally, this consists of a scheduling problem, in which Waze decides the optimal sequence of vehicles from all directions, proceeding based on the data it collects. Then, given the most efficient schedule, Waze optimizes on the speed at which a vehicle approaching the intersection. Travelers could thus save significant amounts of time, since there is no more waiting at intersections. However, when the entire urban traffic system is considered, the problem is challenging both analytically and computationally.

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Appendix

Proof of Proposition 2. To analyze the problem (P2), we first write the dual problem (DP2).

$$
\begin{aligned}
\sup \quad & \phi_0 + \phi_\lambda \\
\text{s.t.} \quad & \left(\frac{D_1 + \lambda + \frac{1}{2}}{s_1} - \frac{D_2}{s_2} \right) \psi u_1 + \phi_\lambda \leq 2 \left(\frac{D_1 + \lambda + \frac{1}{2}}{s_1} \right) \psi, && (13) \\
& \left(\frac{D_1 + \frac{1}{2}}{s_1} - \frac{D_2}{s_2} \right) (1-\psi) u_1 + \phi_0 \leq 2 \left(\frac{D_1 + \frac{1}{2}}{s_1} \right) (1-\psi), && (14) \\
& \left(\frac{D_1 + \lambda}{2s_1} - \frac{D_2 + \frac{1}{2}}{2s_2} \right) \psi u_1 + \left(\frac{D_2}{2s_2} - \frac{D_1 + \lambda + \frac{1}{2}}{2s_1} \right) \psi u_2 + \phi_\lambda \\
& \quad \leq \left(\frac{D_1 + \lambda}{s_1} + \frac{D_2}{s_2} \right) \psi, && (15) \\
& \left(\frac{D_1}{2s_1} - \frac{D_2 + \frac{1}{2}}{2s_2} \right) (1-\psi) u_1 + \left(\frac{D_2}{2s_2} - \frac{D_1 + \frac{1}{2}}{2s_1} \right) (1-\psi) u_2 + \phi_0 \\
& \quad \leq \left(\frac{D_1}{s_1} + \frac{D_2}{s_2} \right) (1-\psi), && (16) \\
& \left(\frac{D_2 + \frac{1}{2}}{s_2} - \frac{D_1 + \lambda}{s_1} \right) \psi u_2 + \phi_\lambda \leq 2 \left(\frac{D_2 + \frac{1}{2}}{s_2} \right) \psi, && (17) \\
& \left(\frac{D_2 + \frac{1}{2}}{s_2} - \frac{D_1}{s_1} \right) (1-\psi) u_2 + \phi_0 \leq 2 \left(\frac{D_2 + \frac{1}{2}}{s_2} \right) (1-\psi), && (18) \\
& u_1, u_2 \leq 0.
\end{aligned}
$$

To further analyze the model, we divide the state space $\{(D_1, D_2) | D_1, D_2 \in \mathbb{N}_+\}$ into five regions. Let the value function be $V_2(D_1, D_2)$.

Region I. $\frac{D_2+\frac{1}{2}}{s_2} \leq \frac{D_1}{s_1}$.

In this region, for any θ, there is no strictly positive $p(1|\theta)$ or $p(2|\theta)$ such that constraint (3) is satisfied. Thus, $p(1|\theta) = p(2|\theta) = 0,\ p(0|\theta) = 1,\ \forall \theta$. Thus, we have

$$V_2(D_1, D_2) = 2 \left(\frac{D_2 + \frac{1}{2}}{s_2} \right).$$

Region II. $\frac{D_1}{s_1} < \frac{D_2+\frac{1}{2}}{s_2} \leq \frac{D_1+\psi\lambda}{s_1}$.

In this region, constraint (3) implies constraint (4). Thus, we can ignore constraint (4). It means that, in the dual problem (DP2), we can set the decision variable u_2 to be 0. Now, (DP2) is transformed into the following problem.

$$
\begin{aligned}
\text{(DP2II)} \quad \sup \quad & \psi \min \Bigg\{ 2\left(\frac{D_1+\lambda+\frac{1}{2}}{s_1}\right) + \left(\frac{D_1+\lambda+\frac{1}{2}}{s_1} - \frac{D_2}{s_2}\right) u_1', \\
& \left(\frac{D_1+\lambda}{s_1} + \frac{D_2}{s_2}\right) + \left(\frac{D_1+\lambda}{2s_1} - \frac{D_2+\frac{1}{2}}{2s_2}\right) u_1', 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right) \Bigg\} \\
& +(1-\psi) \min \Bigg\{ 2\left(\frac{D_1+\frac{1}{2}}{s_1}\right) + \left(\frac{D_1+\frac{1}{2}}{s_1} - \frac{D_2}{s_2}\right) u_1', \\
& \left(\frac{D_1}{s_1} + \frac{D_2}{s_2}\right) + \left(\frac{D_1}{2s_1} - \frac{D_2+\frac{1}{2}}{2s_2}\right) u_1', 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right) \Bigg\} \\
\text{s.t.} \quad & u_1' \geq 0.
\end{aligned}
$$

Notice that

$$
\frac{D_1}{s_1} + \frac{D_2}{s_2} < 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right).
$$

Thus, $p^*(0|0) = 0$. Especially, if $\frac{D_1+\frac{1}{2}}{s_1} > \frac{D_2}{s_2}$, then we have $p^*(1|0) = 1$ and $p^*(2|0) = 0$.

Region III. $\frac{D_1+\psi\lambda}{s_1} - \frac{1}{2s_2} < \frac{D_2}{s_2} \leq \frac{D_1+\psi\lambda+\frac{1}{2}}{s_1}$.

The problem is equivalent to solving the following 2-variable LP.

$$
\begin{aligned}
\text{(DP2III)} \quad \sup \quad & \psi \min \Bigg\{ 2\left(\frac{D_1+\lambda+\frac{1}{2}}{s_1}\right) + \left(\frac{D_1+\lambda+\frac{1}{2}}{s_1} - \frac{D_2}{s_2}\right) u_1', \\
& \left(\frac{D_1+\lambda}{s_1} + \frac{D_2}{s_2}\right) + \left(\frac{D_1+\lambda}{2s_1} - \frac{D_2+\frac{1}{2}}{2s_2}\right) u_1' + \left(\frac{D_2}{2s_2} - \frac{D_1+\lambda+\frac{1}{2}}{2s_1}\right) u_2', \\
& 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right) + \left(\frac{D_2+\frac{1}{2}}{s_2} - \frac{D_1+\lambda}{s_1}\right) u_2' \Bigg\} \\
& +(1-\psi) \min \Bigg\{ 2\left(\frac{D_1+\frac{1}{2}}{s_1}\right) + \left(\frac{D_1+\frac{1}{2}}{s_1} - \frac{D_2}{s_2}\right) u_1', \\
& \left(\frac{D_1}{s_1} + \frac{D_2}{s_2}\right) + \left(\frac{D_1}{2s_1} - \frac{D_2+\frac{1}{2}}{2s_2}\right) u_1' + \left(\frac{D_2}{2s_2} - \frac{D_1+\frac{1}{2}}{2s_1}\right) u_2', \\
& 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right) + \left(\frac{D_2+\frac{1}{2}}{s_2} - \frac{D_1}{s_1}\right) u_2' \Bigg\} \\
\text{s.t.} \quad & u_1', u_2' \geq 0.
\end{aligned}
$$

Region IV. $\frac{D_1+\psi\lambda+\frac{1}{2}}{s_1} < \frac{D_2}{s_2} \leq \frac{D_1+\lambda+\frac{1}{2}}{s_1}$.

In this region, constraint (4) implies constraint (3). Thus, we can ignore constraint (3). Now, (DP2) is transformed into the following problem.

$$
\begin{aligned}
\text{(DP2IV)} \quad \sup \quad & \psi \min \left\{ 2\left(\frac{D_1+\lambda+\frac{1}{2}}{s_1}\right), \left(\frac{D_1+\lambda}{s_1}+\frac{D_2}{s_2}\right)+\left(\frac{D_2}{2s_2}-\frac{D_1+\lambda+\frac{1}{2}}{2s_1}\right)u_2', \right. \\
& \left. 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right)+\left(\frac{D_2+\frac{1}{2}}{s_2}-\frac{D_1+\lambda}{s_1}\right)u_2' \right\} \\
& +(1-\psi) \min \left\{ 2\left(\frac{D_1+\frac{1}{2}}{s_1}\right), \left(\frac{D_1}{s_1}+\frac{D_2}{s_2}\right)+\left(\frac{D_2}{2s_2}-\frac{D_1+\frac{1}{2}}{2s_1}\right)u_2', \right. \\
& \left. 2\left(\frac{D_2+\frac{1}{2}}{s_2}\right)+\left(\frac{D_2+\frac{1}{2}}{s_2}-\frac{D_1}{s_1}\right)u_2' \right\} \\
\text{s.t.} \quad & u_2' \geq 0.
\end{aligned}
$$

Notice that

$$\frac{D_1}{s_1}+\frac{D_2}{s_2}<2\left(\frac{D_2+\frac{1}{2}}{s_2}\right).$$

Thus, $p^*(0|0)=0$. Especially, if $\frac{D_1+\frac{1}{2}}{s_1}>\frac{D_2}{s_2}$, then we have $p^*(1|0)=1$ and $p^*(2|0)=0$.

Region V. $\frac{D_2}{s_2}>\frac{D_1+\lambda+\frac{1}{2}}{s_1}$.

In this region, for any θ, there is no strictly positive $p(0|\theta)$ or $p(1|\theta)$ such that constraint (4) is satisfied. Thus, $p(0|\theta)=p(1|\theta)=0,\ p(2|\theta)=1,\ \forall\theta$. It means that Waze will route every vehicle to Route 1 no matter the state of the traffic. Thus, we have

$$V_2(D_1,D_2)=2\left(\frac{D_2+\psi\lambda+\frac{1}{2}}{s_1}\right).$$